Diurnal Shear Instability, the Descent of the Surface Shear Layer, and the Deep Cycle of Equatorial Turbulence

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ABSTRACT

A new theory of shear instability in a turbulent environment is applied to eight days of velocity and density profiles from the upper-equatorial Pacific. This period featured a regular diurnal cycle of surface forcing, together with a clear response in upper-ocean mixing. During the day, a layer of stable stratification and shear forms at the surface. During late afternoon and evening, this stratified shear layer descends, leaving the nocturnal mixing layer above it. Using high-resolution current measurements, the detailed structure of the descending shear layer is seen for the first time. Linear stability analysis is conducted using a new method that accounts for the effects of preexisting turbulence on instability growth. Shear instability follows a diurnal cycle linked to the afternoon descent of the surface shear layer. This cycle is revealed only when the effect of turbulence is accounted for in the stability analysis. The cycle of instability leads the diurnal mixing cycle, typically by 2–3 h, consistent with the time needed for instabilities to grow and break. Late at night, the resulting turbulence suppresses further instabilities, lending an asymmetry to the mixing cycle that has not been noticed in previous measurements. Deep cycle mixing is triggered by instabilities formed as the descending shear layer merges with the marginally unstable shear of the Equatorial Undercurrent. In the morning, turbulence decays and the upper ocean restratifies. Wind accelerates the near-surface flow to form a new unstable shear layer, and the cycle begins again.

1. Introduction

Nighttime cooling of the sea surface typically initiates a sequence of mixing and mixed layer deepening that is reasonably described by Monin–Obukhov similarity scaling (Shay and Gregg 1986; Anis and Moum 1994), at least over the bulk of the mixed layer away from the surface. This process is complicated only somewhat by winds (Lombardo and Gregg 1989) and surface waves (Anis and Moum 1995; Terray et al. 1996). The mechanisms by which mixed layer deepening is achieved include penetrative convection (Farmer 1975) and, when the wind blows, Langmuir circulation (Li and Garrett 1997) and shear instability associated with trapping of near-inertial shear at the mixed layer base (D’Asaro 1989). The upper-equatorial ocean is unique in that the large-scale current system set up by sustained zonal winds includes an undercurrent of similar magnitude to, and directed opposite to, the surface current. The resulting shear is sufficient to maintain the upper 60 m (or so) at near-critical Richardson number (i.e., Ri) for extended periods of time. Impressed on the mean flow are clear diurnal mixing cycles (Moum and Caldwell 1985; Gregg et al. 1985). The equatorial diurnal cycle is distinguished by the extension of diurnally varying mixing well beneath the direct influence of surface forcing (Moum et al. 1989; Lien et al. 1995). This has been termed the “deep cycle” of equatorial mixing. In this paper we will use a new method of linear stability analysis, combined with in situ measurements from the equatorial Pacific, to show how shear instability can trigger diurnally varying turbulence, both near the surface and in the deep cycle.

Our most recent equatorial measurements (Moum et al. 2009; Inoue et al. 2012) include velocity profiles with a vertical resolution considerably finer than was
previously possible and that show new details of the near-surface currents. In particular, the measurements clearly reveal the evening descent of a shear layer that builds up at the surface during the day and then descends through the late afternoon and evening. It has long been suspected that the development of the nocturnal mixing layer can be initiated by shear instability such as would result from the descending shear layer (e.g., Price et al. 1986). That idea has been applied in simulations of near-surface equatorial mixing (Schudlich and Price 1992), but has not been tested via explicit stability analysis. That analysis is now possible thanks to two new developments: 1) the aforementioned high-resolution velocity profiles and 2) a new instability theory that takes account of the role of turbulence. The new theory (Liu et al. 2012) considers not only the generation of turbulence by instability but also the effect of turbulence on subsequent instabilities. The diurnal cycle of near-surface mixing at the equator turns out to be an excellent natural laboratory for testing this theory.

Extending the stability analysis in the manner of Liu et al. (2012), we find a clearly defined diurnal cycle in shear instabilities near the surface, initiated each afternoon by the descending shear layer. Turbulence is observed to increase a few hours after instabilities are predicted, consistent with the time needed for instabilities to grow and break. Subsequently, shear and stratification decrease because of mixing and, after several hours, Ri exceeds 1/4, a condition sufficient to ensure that disturbances in a nonturbulent, inviscid, and nondiffusive fluid would be damped. However, long before this condition is reached, the rate of detection of new instabilities drops sharply because new instabilities cannot grow in the presence of strong turbulence. The diurnal cycle of instability is found only when this effect is accounted for. These results point to a new understanding of the relationship between shear instability and turbulence in geophysical flows.

The origin of deep cycle turbulence has been the object of considerable interest, due both to its novelty and its practical importance as a mechanism for the exchange of momentum and scalar concentrations such as heat between the surface and the deep ocean (Moum et al. 2013). Hypotheses advanced to date fall into three classes. In the first, deep cycle turbulence is generated locally by shear instability, and its diurnal variability results from some unspecified effect that causes the mean flow at depth to become unstable at 24-h intervals (e.g., Moum et al. 1992). In the second, instability occurs above the depth range of the deep cycle, where it is more susceptible to the influence of the diurnal surface forcing. The normal-mode structure of that instability may extend deeply enough to perturb the flow in the deep cycle region (Sun et al. 1998). Alternatively, the instability may launch downgoing waves that propagate into the deep cycle region and break, possibly as a result of constructive interference between different wave packets (Moum et al. 1992). A variant of this hypothesis is that downgoing waves are launched by some other mechanism such as convective plumes impinging on the base of the mixed layer (e.g., Wijesekera and Dillon 1991).

A third hypothesis has emerged recently through the numerical simulations of Pham and Sarkar (2010) and Pham et al. (2012). In those simulations, shallow shear instabilities launch hairpin vortices downward into the Equatorial Undercurrent (EUC). The vortices interact nonlinearly with the mean shear to tap the kinetic energy of the EUC and produce turbulence. The diurnal aspect has been addressed in a recent contribution by Pham et al. (2013), who find that the initial shear instabilities originate in shallower water during the evening with the relaxation of surface heating. Thereafter, hairpin vortices are launched downward into the EUC, wind-driven momentum is likewise transported downward, and nonlinear interaction with the EUC shear produces the deep cycle.

The results described here support the first hypothesis, that is, that the deep cycle is triggered by local shear instability. Moreover, we will suggest a mechanism for the required diurnal variation of the mean flow, namely the descent from the surface of the daytime shear layer. The deep cycle region is always marginally unstable, and the added shear contributed by the descending shear layer tips it into the unstable state.

We begin in section 2 by describing the observational techniques and the resulting dataset that is the basis for our analyses. We then proceed to test the stability of the observed flows to small perturbations. In section 3, we describe the new method for computation of unstable modes. In preparation for the analyses of the observed profiles, we revisit the classical case of a hyperbolic tangent shear layer (Hazel 1972), but with various profiles of eddy viscosity and diffusivity. The main results are given in section 4, where we carry out the stability analysis on observed profiles. In section 4b, we illustrate the methodology using detailed results for a single profile. Section 4c describes statistics of the full ensemble of 698 unstable modes found over 384 profiles. We show in section 4d that shear instability follows a diurnal cycle, and explore its relationship to the diurnal mixing cycle using a novel phase diagram based on the Reynolds and Richardson numbers that govern instability. In section 4e, we turn to the deep cycle and demonstrate how our results are consistent with its generation by local shear instability triggered by the daily descent of the shear layer. Results are summarized in section 5.
2. Observational context

We begin by describing the observational methodology [for additional details see Moum et al. (2009)] and examining in detail the physical structure of the diurnal mixing layer revealed by these observations.

a. Measurements

The research vessel (R/V) Wecoma maintained station at 0°, 140°W over 16 days in October and November 2008. Measurements of the velocity structure were made using two hull-mounted acoustic Doppler current profilers (300, 75 kHz) and a single 150-kHz unit deployed over the side of the ship. The use of these three units was essential for obtaining high-resolution measurements to as great a range as possible. At the equator, measurement range is limited by the weak acoustic scattering environment. Current profiles were sampled at intervals from 0.5 to 5 s with 2-m vertical resolution to 80-m depth, 4-m resolution to 150-m depth, and 6-m resolution deeper, and the data blended from successively lower-frequency instruments with increasing depth.

Measurements of temperature and conductivity (from which salinity and density were computed) and the turbulent kinetic energy dissipation rate $\epsilon$ were made using the turbulence profiler Chameleon (Moum et al. 1995). Profiling was carried out continuously at the drop rate of 8–10 h$^{-1}$ to 200-m depth. Surface fluxes were estimated using bulk formulae (Fairall et al. 1996) applied to standard ship meteorological measurements.

b. Overview of surface conditions, currents, stratification, and turbulence

The wind stress (Fig. 1a) remained within a factor two of 0.11 N m$^{-2}$, its root-mean-square value. The net

![Fig. 1. (a) Absolute wind stress, (b) net surface heat flux, and (c) sea surface temperature at 0°, 140°W in the equatorial Pacific during an 8-day period in boreal fall 2008. Vertical lines indicate the time used as an example in section 4b.](image)
surface heat flux showed a regular diurnal cycle that was reflected in the sea surface temperature (Figs. 1b,c). On 2 November, a front associated with a tropical instability wave passed the measurement point (sharp temperature jump in Fig. 1c), introducing a significantly different dynamical regime (Moum et al. 2009; Inoue et al. 2012). In particular, wave-generated currents reduced the regularity of the diurnal mixing cycle. Accordingly, the present study focuses on the 8-day period prior to the passage of the front.

The zonal current was dominated by a strong, eastward-flowing EUC focused around 100-m depth (Fig. 2a). Though the trade winds were strong, the zonal current at the surface was near zero. The meridional current, flowing EUC focused around 100-m depth (Fig. 2a). Though the trade winds were strong, the zonal current at the surface was near zero. The meridional current was weak in the mean but highly variable. The EUC coincided with the stable stratification of the seasonal thermocline (Fig. 2b).

The squared shear $U_z^2 + V_z^2$ was strongest on the upper flank of the EUC, as was the stratification $N^2 = B_z$ (Fig. 3), where $B = -g \rho(z) - \rho_0 / \rho_0$ is the buoyancy, $\rho$ is the sorted density with characteristic value $\rho_0$, and subscripts indicate partial derivatives. For the first time at the equator, these high-resolution velocity measurements clearly show the diurnal evolution of the near-surface shear. Beginning each afternoon, a layer of concentrated shear descended from the surface, accompanied by a layer of stable stratification (Figs. 3b,c). The resulting stratified shear layer reached depths of 40–60 m before dissipating in the late evening.

The turbulent kinetic energy dissipation rate $\epsilon$ (Fig. 3d) was strong in a layer close to the diurnally descending stratified shear layer. The diurnal signal in $\epsilon$ extended below the base of the surface mixing layer to a maximum depth of 50–110 m. This is the deep cycle of equatorial mixing (Moum et al. 1989; Lien et al. 1995).

c. The diurnal mixing cycle

To better illustrate the diurnal character of equatorial upper-ocean mixing as observed in this experiment, we construct a canonical day (Fig. 4). This is done by averaging over every occurrence of a given hour to form a typical flow profile for that time of the day. We refer to this as the phase average, and denote it, for any quantity $x$, as $\langle x \rangle$. In the cases of the squared shear $\langle S^2 \rangle = \langle U_z^2 + V_z^2 \rangle$, the squared buoyancy frequency $\langle B_z \rangle$, and the turbulent kinetic energy dissipation rate $\langle \epsilon \rangle$, the average is a geometric mean. In the case of $\langle S^2 \rangle$, the order of operations is important; the squared shear is computed for individual 30-min profiles before the average is taken. A gradient Richardson number (i.e., $Ri$), formed as the ratio of the diurnally averaged $N^2$ and $S^2$, provides a rough indication of the potential for shear instability (e.g., Miles 1961).

The phase-averaged surface buoyancy flux $\langle J_b \rangle$ shows a distinct diurnal cycle, heating between 0700 and 1700 local time and cooling in the remaining hours (Fig. 4a). The wind stress (not shown), shows no such cyclic behavior. Despite its absence in the wind stress, a diurnal cycle is clearly evident in the near-surface squared shear (Fig. 4b). A layer of enhanced shear forms in the morning (upper right), reaching 15-m depth around 1400 LT. Its amplitude reaches a maximum of $2 \times 10^{-4}$ s$^{-2}$ at the same time, and that maximum begins to descend around 1500 LT at an average rate of 6 m h$^{-1}$ (white curve beginning near upper left). As this maximum passes any given depth, the shear there continues to increase in time, so that the maximum with respect to time (dashed curve) occurs up to two hours later than the maximum with respect to depth (solid curve). By about 2200 LT, the shear maximum reaches 60-m depth, after which it is no longer clearly defined. As the maximum is descending, the near-surface shear decreases by two orders of magnitude.

The squared buoyancy frequency (Fig. 4c) follows a pattern similar to that of the shear. The maximum with respect to depth coincides with the shear maximum (white curve). However, the stratification does not continue to increase after the shear maximum passes, but instead decays quite quickly.

In a thin layer adjacent to $z = -10$ m (the upper limit of our measurements), $Ri$ is highly variable, with values less than the nominal critical value $1/4$ occurring at night (black contour just visible at the top of Fig. 4d). Below $z = -12$ m is a layer of thickness 2–20 m in which $Ri$ is $>1/4$ (shaded from yellow to red). Deeper down is a layer of thickness 30–40 m in which $Ri$ is $<1/4$ throughout the diurnal cycle (shaded blue). The lowest $Ri$ values appear near the same time as the descending shear...
maximum and persist for a few hours afterward (dark blue), reflecting the previously noted fact that shear remains after stratification has mixed out.

The turbulent dissipation rate (Fig. 4e) shows a clear diurnal cycle similar to that seen previously at this location (e.g., Gregg et al. 1985; Moum and Caldwell 1985; Moum et al. 1989; Peters et al. 1994). The phase-averaged $\epsilon$ is small during the day (except above 12 m, where measurements may be affected by ship wake), with values less than $10^{-7}$ W kg$^{-1}$ in a layer extending to about 35 m. As the shear layer descends, $\epsilon$ begins to increase. Its temporal maximum at any given depth does not coincide with that of the shear, however, but rather occurs 2–3 h later.

We know from direct numerical simulations that shear instability in its initial growth phase is only weakly dissipative; dissipation becomes strong only after the billows grow to finite amplitude and break (Smyth and Moum 2000b,a; Smyth et al. 2001). This is also consistent with the observations shown in Moum et al. (2011, especially cases I and II in their Fig. 8), which suggest narrowband oscillations growing and culminating in a burst of turbulent dissipation. Therefore, we propose that the delay between peak shear and peak $\epsilon$ is the time needed for shear instabilities, initiated at the time of the shear maximum, to grow and break.

Diurnally enhanced dissipation rates are evident down to about 60-m depth. The subsequent decay of this turbulence is quite rapid (2–3 h) in a layer around 15–25-m depth, but takes several hours at deeper depths. This longer decay time below $z \sim -25$ m coincides with a relatively large maximum in $\langle S^2 \rangle$ and a lower minimum in $R_i$.

### 3. Linear stability analysis of turbulent, stratified shear flow

Results from the previous section reinforce the idea (e.g., Price et al. 1986; Schudlich and Price 1992; Pham and Sarkar 2010) that the diurnal descent of the wind-driven mixed layer is driven by shear instability. In what follows, we will test that hypothesis further by means of an explicit linear stability analysis of the observed flows. With this analysis, we will show that in the late afternoon and evening, $R_i$ is low enough to allow instabilities to grow, initiating the sequence of events that leads to diurnal mixing.

Our linear analysis is designed to detect the instabilities incipient in the measured profiles and to predict their
spatial and temporal scales and structures under a variety of conditions. A weakness is that it applies only to the initial growth phase of the instabilities, providing no information about the later nonlinear evolution and breakdown into turbulence. Moreover, it takes no account of air–sea fluxes, nor of internal waves too small or fast to be resolved in the observations, except for those generated by the instabilities themselves.

Fig. 4. Diurnal phase averages based on 8 days of observations. The abscissa is the LT, which lags UTC by nine hours. (a) \(J_b\), (b) \(S^2\). (c) Four times the squared buoyancy frequency \(4N^2\). (d) The ratio \(\frac{N^2}{S^2}\). The black contour represents \(Ri = \frac{1}{4}\). Colors are mapped so that the region around \(Ri = \frac{1}{4}\) is white. (e) \(\frac{1}{C_15}\). In (b)–(e), the solid white curve indicates the shallowest local max \(S^2\) with respect to depth. In (b), the dashed curve shows the first local max with respect to time. White horizontal lines indicate \(z = -20\) and \(-56\) m for future reference.
Starting with a mean flow defined by 30-min, 2-m-averaged profiles of \( U, V, B, \) and \( \epsilon \), we investigate the growth of infinitesimal perturbations. The development of the method has been described in Moum et al. (2003) and Smyth et al. (2011), and is given in its present form by Liu et al. (2012). In this section, we give a brief review and discuss a few novel aspects. We also build intuition for the analysis of the relatively complex observed profiles by looking at the simple case of a hyperbolic tangent shear layer with various profiles of eddy viscosity and diffusivity.

### a. Methods

We imagine the flow as consisting of three components. The first is the mean flow; the second is a small-amplitude disturbance having the normal-mode form

\[
\Phi' = \Phi(z) \exp[\alpha t + i(kx + ly)],
\]

where \( \Phi \) represents any perturbation quantity, and \( \Phi \) is its depth-dependent complex amplitude. The constant \( \alpha = \sigma - i\omega \), where \( \sigma \) is the exponential growth rate and \( \omega \) is the angular frequency. The zonal and meridional wavenumbers are \( k \) and \( l \), respectively. Useful derived quantities include

- the wave vector magnitude \( \kappa = \sqrt{k^2 + l^2} \);
- the wavelength \( \lambda = 2\pi/\kappa \);
- the azimuthal angle measured counterclockwise from east \( \phi = \tan^{-1}(l/k) \);
- the horizontal phase speed \( c = -\sigma/\kappa \); and
- the critical level \( z_c \), defined by \( \hat{U}(z_c) = c \), where \( \hat{U} = (kU + lV)/\kappa \) is the horizontal velocity in the direction of the wave vector.

The third component of the flow, newly added to the theory, is a field of small-scale turbulence, perhaps resulting from an energy cascade initiated by some previous instability. We assume that the ambient turbulence is focused on an energy cascade initiated by some previous instability. Both boundaries therefore have vertical velocity, vertical shear of horizontal velocity, and buoyancy perturbation fixed at zero. As explained below, the lower boundary is overlain by a sponge layer to absorb any downward energy flux from the instabilities. Derivatives are discretized using second-order centered differences. The result is a linear generalized eigenvalue problem that can be solved numerically using standard methods.

A factor not considered here is the feedback of instability on the eddy coefficients. This can cause a separate class of layering instabilities that are potentially quite significant (e.g., Phillips 1972; Posmentier 1977; Smyth and Ruddick 2010; Smyth et al. 2012). This possibility will be pursued in a separate study.

### b. Results for a set of idealized models

Before commencing the analysis of the equatorial Pacific data, we consider the general effects of \( A(z) \) and \( K(z) \) in a set of simple models. A parallel, stratified shear layer approximates the background flow profiles that produce shear instability in the equatorial ocean (Smyth et al. 2001, 2011):

\[
\frac{U(z)}{\Delta U} = \frac{B(z)}{\Delta B} = \tanh \frac{z}{h},
\]

where \( h \) is the half-thickness of the layer, and \( \Delta U \) and \( \Delta B \) are the half-changes in velocity and buoyancy (Figs. 5a,b). For these idealized examples, the boundaries are placed at \( z = \pm 8h \). Nondimensionalized parameters are defined using \( h \) and \( \Delta U \) as scales for length and velocity, respectively.

The stability of (2) is governed by three scalar parameters: the bulk Richardson number \( \text{Ri}_b = h\Delta B/\Delta U^2 \) (which is also the minimum value of the gradient Richardson number \( B_2/S^2 \), the Prandtl number \( A_\omega/K_\omega \), and the turbulence Reynolds number

\[
\text{Ri}_b = \frac{\Delta U h}{A_\omega},
\]

where \( A_\omega \) and \( K_\omega \) are the eddy viscosity and diffusivity at the shear maximum \( z = 0 \), as well as by the vertical structures of \( A_\omega \) and \( K_\omega \).

In geophysical flows, the mixing coefficients \( A_\omega \) and \( K_\omega \) vary widely with depth. Therefore, it is interesting to explore some idealized models of variable \( A_\omega \) and \( K_\omega \). For this initial exploration, we set \( K_\omega = A_\omega \) and choose

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1 The coefficients \( A_\omega \) and \( K_\omega \) describe vertical mixing only. Although horizontal mixing may be important, efforts to parameterize it are at a relatively early stage (Liu et al. 2012). We therefore defer consideration of horizontal mixing for a future project.
the following four models for $A_y$, representing four basic forms of variability that might be encountered in oceanic shear layers:

$$\frac{A_y}{A_{y0}} = \frac{K_y}{K_{y0}} = \begin{cases} 1 & \text{(model 1)}, \\ 1 + \tanh \frac{z}{h} & \text{(model 2)}, \\ \text{sech}^2 \frac{z}{h} & \text{(model 3), and} \\ 2 - \text{sech}^2 \frac{z}{h} & \text{(model 4)}. \end{cases}$$

Note that the value at $z = 0$, the center of the shear layer, is the same for all models. In model 1, $A_y$ is uniform. In model 2, $A_y$ is larger above the shear layer and smaller below. (In the supposed Boussinesq flow, results extend by symmetry to the opposite case where viscosity is smaller above the shear layer and larger below.) In model 3, $A_y$ is proportional to the shear; hence, it is concentrated in the shear layer and drops to zero both above and below. In model 4, the reverse is true: the shear layer represents a region of reduced eddy viscosity.

We begin by exploring the effects of finite eddy viscosity in the simple case of an unstratified shear layer. We extend the results of Betchov and Szewczyk (1963), who examined the special case of uniform viscosity (our model 1). At large $Re_h$, the four eddy viscosity models give nearly the same growth rate, phase velocity, and wavelength (Figs. 6a,b,c, respectively). These high values of $Re_h$ approximate the inviscid limit, in which the vertical form of the eddy coefficients is irrelevant. As $Re_h$ is reduced (equivalent to increasing $A_{y0}$), differences become evident. The growth rate of the fastest-growing mode (Fig. 6a) diminishes markedly as $Re_h$ is reduced. That reduction is greatest for model 4 and is least for model 3. The phase velocity (Fig. 6b) is zero for models 1, 3, and 4, regardless of $Re_h$, as expected from symmetry. In model 2, however, the symmetry is broken.
...and the phase velocity of the fastest-growing mode becomes increasingly negative as Re_h is reduced. This indicates that the critical level (where the phase velocity matches the mean flow) is located below the center of the shear layer, where the mean flow velocity is negative. This makes intuitive sense, because turbulent damping is reduced in the lower half of the shear layer, and the region is in this respect more hospitable to growth. For each model, the preferred wavelength (Fig. 6c) increases with decreasing Re_h.

We next extend the analysis to stably stratified shear layers. The case of uniform coefficients (similar to our model 1) with turbulent Prandtl number \( Pr_t = 1 \) and various models of \( A_y(z) \) as indicated in the legend. Re is computed using both \( h \) (solid) and \( \lambda \) (dashed). Ri_b is equal to the min gradient Richardson number. The critical value of \( Ri_b \) becomes less as Re decreases or as the eddy viscosity, a measure of turbulence, increases. The large asterisk corresponds to the example shown in Fig. 8.

In the analysis of observed shear layers, it is usually straightforward to identify a value for Ri_b such that the comparison with linear theory (e.g., Maslowe and Thompson 1971) is meaningful: one simply uses the local values of \( N^2 \) and \( S^3 \) at the shear maximum. In contrast, Re_h is an intrinsically nonlocal parameter whose computation requires identification of the thickness of the shear layer and the velocity change across it, and there is inevitably some arbitrariness in this. To avoid this problem, we define an alternative version of the Reynolds number that is straightforward to compute observationally by using the mode wavelength \( \lambda = 2\pi/k \) in place of \( h \) as the length scale:

\[
Re_\lambda = \frac{S_0(\lambda/4\pi)^2}{A_{40}},
\]

where \( S_0 \) is the maximum shear magnitude. This Reynolds number characterizes not only the mean flow but also the particular mode in question, a property that we will find useful in the analyses to follow. The factor \( 4\pi \) ensures that \( Re_\lambda = Re_h \) over a large and important range of parameter space, specifically the double limit Re \( \rightarrow \infty \), Ri \( \rightarrow \frac{1}{4} \), where the fastest-growing instability has...
wavelength $\lambda = 4\pi h$ (Holmboe 1962). The second limit can be relaxed without changing this result much: the fastest-growing wavelength in the inviscid case changes by less than 10% as $R_i$ is reduced from $1/4$ to zero. The first limit, $Re \to \infty$, is also a good approximation for geophysical flows in which turbulence has not yet developed or has subsided. However, at small $Re$, the wavelength of the fastest-growing mode increases, because larger-scale motions are less vulnerable to viscous damping (Fig. 6c). As a result, $Re_\lambda > Re_b$ for modes in highly viscous (or turbulent) shear layers.

The black, dashed curve marked “$Re_\lambda, 1”$ on Fig. 7 is the stability boundary for model 1 computed using $Re_\lambda$ in place of $Re_b$. To obtain this curve, we first compute the stability boundary in terms of $Re_b$ (black, solid curve) by choosing successive values of $Re_b$ and, for each value, finding the maximum $R_i$ at which instability can occur. The corresponding wavelength is then used to determine $Re_\lambda$.

When eddy coefficients vary in the vertical, the stability boundary is shifted (blue, red, and green curves on Fig. 7). In models 2 and 3, the unstable regime is expanded relative to model 1, with the effect being most pronounced in model 3. This suggests that instabilities are structured so as to take advantage of reduced viscosity away from the center of the shear layer. This occurs below the shear layer for model 2 (as noted previously) and both above and below for model 3. In model 4, the unstable regime is contracted relative to model 1. This is consistent with the fact that viscosity increases away from the center of the shear layer. In all models, $Re_\lambda > Re_b$.

We next examine the vertical structure of a typical mode from the unstable regime of model 1. The mode we have chosen is identified by the asterisk on Fig. 7. The case has Richardson and Reynolds numbers characteristic of an observed instability to be described in detail in section 4b. It has nondimensional growth rate of 0.11, phase velocity of zero, and wavelength of 14.3. The vertical structure (Fig. 8) shows a strong peak in perturbation kinetic energy surrounding the critical level $z = 0$ and secondary maxima near $z = \pm 1.5$. The upward momentum flux $u'w'$ and downward buoyancy flux $b'w'$, both focused at $z = 0$, are consistent with a disturbance that grows by extracting kinetic energy from the mean flow and does work against gravity in the process. The pressure-driven energy flux $p'w'$ carries perturbation kinetic energy vertically both up- and downward from $z = 0$ and deposits it in the outer regions of the shear layer, accounting for the secondary maxima in the kinetic energy profile.

4. Application to measured profiles

a. Methods

Eddy coefficients are often estimated from microstructure observations using standard forms based on the equilibrium turbulent kinetic energy balance (Osborn 1980):

$$A_v = (1 + \Gamma) \frac{\epsilon}{N^2} \quad \text{and} \quad (6)$$

$$K_v = \frac{\Gamma \epsilon}{N^2}, \quad (7)$$

where $\Gamma$ is taken to be 0.2. Computation of $K_v$ from (7) is often complicated by locally small, or even negative, values of the denominator $N^2$. Here, we use a variation based on the result that the $Pr_t = A_v/K_v$ can be closely approximated as a linear function of $R_i$:

$$Pr_t = Pr_0 + aR_i \quad (8)$$

[see Esau and Grachev (2007) for a survey]. For the constants in (8), we adopt the values derived by Zilitinkevich et al. (2006): $Pr_0 = 0.8$ and $a = 5$. Substituting $K_v = A_v/Pr_t$ into the equilibrium turbulent kinetic energy balance $A_v S^2 - K_v N^2 = \epsilon$, we obtain

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4 The curves for models 2, 3, and 4 were not calculated below $R_i = 0.08$ because of resolution constraints. Our algorithm is optimized for computation of unstable modes from observational data and is inefficient for identifying stability boundaries. Also, longwave instabilities (e.g., Defina et al. 1999) complicate the results at low $Re$. Further study is needed to establish the relevance of these modes for the upper ocean.
\begin{align}
A_v &= \frac{Pr_t}{Pr_s - Ri S^2} \epsilon \\
K_v &= \frac{A_v}{Pr_f} \tag{9}
\end{align}

With \( Pr_t \), approximated by \( (8) \), \( A_v \) and \( K_v \) are very similar to the values given by \( (6) \) and \( (7) \) at large \( Ri \), but \( K_v \) remains finite as \( Ri \to 0 \). The instability of \( (7) \) is now ameliorated, because the denominator of \( (9) \) is positive provided only that \( N^2 > -0.25^2 \) (with the present parameter values), a condition usually satisfied in practice even when \( N^2 \approx 0 \). The alternative approach also replaces the assumption \( \Gamma = 0.2 \) with \( (8) \), which we would argue is in better agreement with recent studies [compare Esau and Grachev (2007) with, e.g., Moum (1996), Ruddick et al. (1997), and Smyth et al. (2001)].

For application to the present observational data, 2-m-averaged profiles are interpolated to 0.5-m spacing using cubic splines. Only measurements between \( z = -10 \) and \( -120 \) m are used. In the upper 10 m, \( U, V, B, A_v, \) and \( K_v \) are extrapolated using polynomial fits such that the first derivative is continuous at \( z = -10 \) m and goes to zero at \( z = 0 \). A free-slip, impermeable lower boundary is placed at \( z = -160 \) m, and the mean profiles between there and \( z = -120 \) m function as a sponge layer, having minimal effect on disturbances above \( z = -120 \) m and damping downgoing radiation before it can reflect. To accomplish this, profiles are extrapolated using polynomial fits as above, with the additional provision that \( A_v \) and \( K_v \) increase from the observed values at \( z = -120 \) m to \( 0.1 \text{ m}^2\text{s}^{-1} \) at \( z = -160 \) m. Analysis of the resulting profiles yields unstable modes with critical levels located between \( z = -10 \) and \( -120 \) m, although only those shallower than \( z = -75 \) m are retained as explained below.

The stability analysis is repeated over a polar grid of wave vectors. For the present application, 23 values of \( \kappa \) are tested, ranging from \( 2\pi/7 \) to \( 2\pi/1000 \) m. The azimuthal angle (i.e., \( \phi \)) ranges from \(-90^\circ \) to \( 90^\circ \) in \( 10^\circ \) increments. For a given wave vector \( (\kappa, \phi) \), the number of unstable modes ranges from zero to about 10. These represent different mode families, usually focused in different depth ranges (e.g., Sun et al. 1998) and each having its own fastest-growing mode. Identification of mode families is automated using the fact that, while most mode properties vary considerably with \( \kappa \) and \( \phi \), the critical level \( z_c \) is relatively consistent. A histogram of \( z_c \) is constructed and peaks identified. All modes close to a given peak (i.e., having critical levels between the adjacent minima of the histogram) are considered part of the same mode family. For each mode family, the fastest-growing mode is identified and retained for further analysis.

Five criteria are adopted to reject modes that are likely to be unphysical.

1) The critical level must lie below \(-10 \) m to minimize the effects of the extrapolated region between this depth and the surface.
2) Because the theory treats the mean flow as stationary, we require that modes grow on time scales faster than that on which the mean flow evolves. Application of this criterion depends on the averaging operation used to define the mean flow. As defined here, the mean flow is not allowed to vary faster than the Nyquist frequency \( 1 \text{ h}^{-1} \). For consistency, we therefore require that growth rates exceed \( 1 \text{ h}^{-1} \).
3) Instabilities of inviscid, nondiffusive, stratified shear layers typically have wavelength around \( 2\pi \) times the thickness of the shear layer (Hazel 1972; Smyth et al. 2011). Based on this, modes with \( \lambda < 25 \) m are likely to grow from layers of thickness \(<4 \) m, the Nyquist wavelength of the 2-m vertical bins defining the mean profile, and may therefore be unduly influenced by the spline interpolation. We discard modes with \( \lambda < 25 \) m on this basis.
4) Any mode family must include at least one resolved mode with a larger wavelength than the fastest-growing mode, and at least one with smaller wavelength. This effectively rejects modes whose true maxima lie outside the range of wavelengths tested. The wavelength range is chosen to make this rare.
5) Small variations in the measured velocity due to the blending of the two ADCP profiles (section 2a) can create spurious instabilities. We have found that all such spurious modes have critical levels deeper than \( 75 \) m, so we exclude those modes from the present analysis.

b. Detailed analysis of a characteristic profile

We begin with a close look at a sample profile obtained at 2245 UTC (1345 local time) 29 October 2008 (vertical lines on Fig. 3). This profile yielded five unstable modes that passed the selection criteria. We will consider all five, but our main focus is on the shallowest as it coincided with the descending shear layer.

In the early afternoon of 29 October, the EUC was strong, generating intense zonal shear, while meridional currents were weak (Fig. 9a). A near-surface shear layer had formed and begun to descend (Fig. 9b, indicated by the horizontal line labeled “1”). The fact that \( S^2 \) exceeded \( 4N^2 \) at this depth shows that \( Ri \approx 1/4 \).

Turbulence was strong enough to produce eddy viscosity near \( 10^{-3} \text{ m}^2\text{s}^{-1} \) at this level (Fig. 9c). Despite this turbulence, an unstable mode was detected. Its critical
level coincided with the shear maximum at 13-m depth. The vertical displacement (Fig. 9d, thick curves) due to this mode showed a thin maximum there, as well as a phase shift (dashed curve) of 0.7π in the direction consistent with energy extraction from the mean shear. This mode had growth rate 1.6 h⁻¹ and wavelength 66.8 m. The wave vector was purely zonal, that is, φ = 0. The Richardson number at the shear maximum was about 0.12, and Reₐ was 500 (asterisk in Fig. 7). The vertical structure of mode 2 is similar to that of mode 1 even though the stratification was a minimum at the critical level (dashed curve in Fig. 9b).

Looking again at the shear profile (Fig. 9b), we note a general increase downward to the EUC core, with several distinct maxima at which Ri < 1/4. In total, five unstable mode families were identified, four of which coincided with shear maxima. The critical levels are marked by horizontal lines in Figs. 9b and 9c and numbered 1–5. Note that mode 3 does not coincide with a shear maximum. The results of section 3b (model 2) suggest that the mode was shifted downward because the eddy viscosity decreased in that direction (Fig. 9c). The displacement eigenfunction for mode 2 (thin curves in Fig. 9d) is similar to mode 1 but focused at a deeper critical level.

The shear profile (Fig. 9b, solid) also features several maxima with Ri < 1/4 where unstable modes were not detected, and it is instructive to ask why. The maximum at 17.5 m (marked 1a), had relatively weak shear. Hence, although Ri was less than 1/4, there was not enough

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**Fig. 9.** Results of stability analysis for a sample profile: a 30-min average centered at 2245 UTC (1345 LT) 29 Oct 2008. (a) Zonal (solid) and meridional (dashed) velocity components. (b) S² (solid) and 4N² (dashed). Horizontal lines marked 1–5 indicate critical levels of the fastest-growing modes of five mode families. Dotted lines marked 1a–3a indicate shear maxima with Ri < 1/4 where unstable modes were not found. (c) Turbulent eddy viscosity (solid) and diffusivity (dashed). Horizontal lines are as in (b). (d) Amplitude (solid) and phase (2π)⁻¹ (dashed) of vertical displacement eigenfunctions for modes 1 (thick) and 2 (thin). Eigenfunctions are normalized so that the max displacement amplitude is 1.
kinetic energy available in the mean flow to drive a mode that would pass the growth rate criterion. This was not the case for the maximum at 33 m (2a), where the shear and stratification levels are very similar to those of the unstable mode 2. The only difference is that the eddy coefficients $A_y$ and $K_y$ are larger than those of mode 2 by about a factor of 2. The maximum at 41 m (marked 3a) exhibited strong shear but also strong stratification, so that $\text{Ri}$ was not much less than 1/4. This mode may also have been damped by preexisting turbulence, because $A_y$ and $K_y$ were unusually large at that depth.

Vertical mode structure is described more fully using profiles of perturbation kinetic energy, momentum flux, buoyancy flux, and energy flux as in section 3b (Fig. 8). For mode 1, the profiles are very similar to those calculated using a weakly stratified, hyperbolic tangent shear layer [Figs. 10a,b; compare with Fig. 8; also Smyth and Peltier (1991, their Figs. 5e–h)]. The perturbation kinetic energy is concentrated near the critical level (Fig. 10a), but is also substantial for about another 10 m both above and below the main peak. The buoyancy flux (Fig. 10b, solid) is negative, indicating that the mode does work against gravity to grow. The momentum flux (dotted) is positive, as the mode fluxes westward momentum from the surface down toward the east-flowing EUC and vice-versa. The pressure–velocity correlation (dashed) fluxes energy both up- and downward from the critical level, explaining the nonzero kinetic energy signals above and below the mean peak. These properties support our identification of the observed instabilities with the classical Kelvin–Helmholtz (KH) instability, a similarity that we find useful in predicting instability wavelengths.

The main difference in vertical structure between the observed example (Fig. 10) and the hyperbolic tangent model (Fig. 8) is the tendency for the energy flux to reach far above and below the shear layer. This indicates a transfer of energy over a thick layer of the water column via wavelike motions. For this example, upward radiation is likely to be imprecise as it is influenced directly by the extrapolation of the background properties from $z = -10$ m to the surface (section 3a). In contrast, the downward radiation is primarily determined by the flow properties below the shear layer, and those are measured directly.

This stratification asymmetry was noted by Smyth et al. (2011) in composite profiles of shear and stratification surrounding instability critical levels, and confirms the relevance of the profiles used the simulations of Pham et al. (2009) and Pham and Sarkar (2010). Pham et al. (2012) have suggested the asymmetry could lead to one-sided Holmboe instabilities (Lawrence et al. 1998; Carpenter et al. 2007), which could in turn spawn KH modes as a secondary instability. Similarly asymmetric profiles (though extended to include the whole undercurrent) were used by Smyth and Moum (2002), who found a radiating mode capable of carrying energy over large vertical distances. None of these departures from the hyperbolic tangent model affect the property of use to us here, namely the length scale of the KH instability.

c. Statistics of scalar properties of 698 modes from 384 profiles

The stability analysis described in section 4a was repeated for 30-min-averaged profiles covering 8 days for a total of 384 profiles, and 698 unstable modes were found that satisfied the selection criteria. A histogram of the critical levels is shown in Fig. 11, along with the mean flow profile for reference. The computed instabilities lie within the sheared zone between the surface and the EUC. Recall that modes with critical levels below $z = -75$ m are excluded in accordance with criterion 5 above. Growth rates range from 1 to 10 h$^{-1}$, with a median of 2.5 h$^{-1}$ (Fig. 12a). The smallest growth rates are the most common. Both the cutoff at $\sigma = 1$ h$^{-1}$ and the median value are influenced by our mode selection criterion 2.

Wavelengths are unimodally distributed and have a median of 104.8 m (Fig. 12b). This is smaller by a factor of 2 than what was found previously by Moum et al. (2011) and Smyth et al. (2011) in the same location (though in different conditions of shear and stratification). This is mainly because, with improved vertical resolution provided by the 2008 dataset, we are able to include the thin shear layers, and the resulting short-wavelength instabilities, that exist in and around the
diurnal mixed layer. In the previous analyses of Smyth et al. (2011), these were classified as "shallow" modes and omitted from further consideration due to insufficient resolution. The azimuthal angle (Fig. 12c) is most commonly close to zero (i.e., the wave vector is directed zonally), but ranges from $-60^\circ$ to $60^\circ$. Absolute frequencies have a median of 0.0025 cps. This is larger than found by Moum et al. (2011) and Smyth et al. (2011), consistent with the fact that the wavelengths of the modes considered here are generally shorter.

The azimuthal angle is almost always within $20^\circ$ of the direction of the shear vector, $\tan^{-1}V_z/U_z$ (Fig. 13), and the medians agree to within $2^\circ$. Thorpe (2012) and Smyth and Thorpe (2011) have investigated the prospects for measuring shear-driven overturns with gliders moving diagonally through the billow train, focusing on the effect of the angle of the glider trajectory relative to the orientation of the billows. We see that, in a veering flow, the latter angle is determined not by the direction of the mean current but by the direction of its vertical derivative, which may be very different.

Growth rates, scaled by the shear (Fig. 14a), lie near or below the values for the inviscid, hyperbolic tangent shear layer. The expected downward trend with respect to $Ri_\kappa$ is evident. The tendency for points to lie below the inviscid curve is a manifestation of damping due to nonzero viscosity and diffusivity. The few growth rates that lie above the curve can be accounted for by differences in the details of the velocity profiles.

Over 98% of the computed modes lie within the stability boundary for the hyperbolic tangent shear layer with uniform viscosity and Prandtl number unity (Fig. 14b; also see Fig. 7 and the accompanying discussion). Moreover, points lying farther from the curve tend to have higher nondimensional growth rates as indicated by the colors. A few instabilities have $Ri_\kappa > \frac{1}{4}$; these are cases where the critical level is not a minimum of $Ri$, and there is a nearby region where $Ri < \frac{1}{4}$; hence the Miles–Howard criterion for instability of inviscid flow is not violated. There are also a few modes that lie outside the stability boundary on the low-$Re_A$ side. This is likely an expression of the fact that the observed $A_y$ and $K_y$ are neither uniform nor equal as assumed in the computation of the stability boundary (see Fig. 7 and accompanying discussion). Additional longwave instabilities due to surface proximity (e.g., Hazel 1972; Defina et al. 1999) could also be a factor. Despite these complications, Fig. 14b suggests that the stability boundary for the hyperbolic tangent shear layer with uniform viscosity and Prandtl number unity provides a useful indication of the stability of real profiles, a fact that will be of central importance in what follows.

FIG. 11. (a) Time-averaged zonal velocity profile. (b) Histogram of critical level heights for an ensemble of 698 detected unstable modes obeying the selection criteria set out in section 4a. Horizontal dashed lines delineate the layer in which critical levels are selected.
d. The diurnal cycle of instability

Critical levels of computed modes (Fig. 15b) occur irregularly at all depths and times of the day, but they have a tendency to cluster near the descending shear maximum, and hence to exhibit a diurnal cycle. The cycle is more clearly visible in a plot of the number of modes found in each hour (Fig. 15c).

The diurnal cycle is most evident nearest the surface. A histogram of modes with critical levels between 75- and 50-m depth does not show the cycle (white bars on Fig. 15c), whereas the cycle is clear in modes above 50 m (gray bars). The mode count is highest during the day, and drops after the surface buoyancy flux changes sign (approximately sunset; Fig. 15a). The mode count remains low through the night and then recovers near sunrise.

The phase relationship between the mode count and the surface buoyancy flux, with shear instability occurring more frequently during the day than at night, could be taken to contradict the hypothesis that shear instability triggers nocturnal mixing. What this result actually reveals is the other half of the story, namely that mixing damps instability. To demonstrate this, we conducted an auxiliary set of analyses focused on a 2-day period (Fig. 16). Over this interval, the analysis was repeated with parameterized turbulence removed, that is, modes were computed using the inviscid and nondiffusive Taylor–Goldstein equation. With turbulence (empty bars with thick borders), the mode count revealed a diurnal cycle just as in Fig. 15. In the nonturbulent case (filled bars), the diurnal cycle was absent. These results suggest that, at least for the bulk of the cases considered here, turbulence acts to suppress instability, and does so mainly at night when turbulence is strongest (Fig. 4e).

The temporal relationship between diurnally varying instability and the near-surface-mean flow (defined by phase-averaged shear, stratification, and turbulence) is illustrated in Fig. 17. For this demonstration, the phase-averaged fields are also averaged over the layer $-25 < z < -15$ m. The Richardson number of the phase-averaged flow first dropped below $\frac{1}{4}$ (i.e., $\langle S^2 \rangle > 4 \langle N^2 \rangle$) near 1500 local time (Fig. 17a). The number of instabilities detected in the layer jumped from 4 between 1400 and 1500 LT to 10 between 1500 and 1600 LT (Fig. 17b). The shear peaked shortly after this, as did the mode count.
The turbulent dissipation rate $h/C_{15}$ (Fig. 17c, solid curve) was relatively small during this time, but it subsequently rose and attained a maximum between 1900 and 2100 LT. As we suggested in section 4d, this time lag may represent the time needed for instabilities to reach their dissipative phase. This idea cannot be tested using linear theory as the time for breaking depends heavily on the form of the initial disturbance. We note, though, that growth over a time scale of hours is consistent with the magnitude of the computed growth rates (Fig. 12a).

As the dissipation rate peaked, the mode count dropped rapidly despite the fact that the Richardson number remained less than 1/4. This decrease in instability count is due to the increase of $h/C_{15}$ and hence of eddy viscosity and diffusivity (section 4a). Over the next several hours, $h/C_{15}$ decayed steadily (as did shear and stratification, though at different rates). In the morning, shear and stratification began to increase again, as did the unstable mode count.

The surface buoyancy flux (Fig. 17c, dashed curve divided by 10) retained a nearly constant value $10^{-7}$ W kg$^{-1}$ throughout the night. In this depth range ($-25 < z < -15$ m), $\langle \epsilon \rangle$ exceeded $\langle J_b \rangle$ by an order of magnitude, showing that convection played only a minor role in promoting turbulent dissipation. Unlike the case of a convective mixing layer, where $\epsilon \approx J_b$ and would therefore have remained constant, $\langle \epsilon \rangle$ peaked in the evening (a few hours after the episode of shear instability as noted above) and then declined steadily.

As we have seen, the diurnal cycle of instability depends on the combination of shear, stratification, and turbulence. We now seek to quantify this dependence by locating the mean flow on the Richardson–Reynolds number plane, as was done for the hyperbolic tangent model flow in section 3b. Again, we consider the phase-averaged flow in a 10-m-thick layer surrounding $z = -20$ m (upper horizontal lines on Figs. 4b–e). The Richardson number is easily defined, as in Fig. 4d. The mode Reynolds number $Re_l$ can also be defined, provided we can specify a value for $l$. For this, we use 50 m, the median wavelength of unstable modes whose critical levels lie in the range $-25 < z < -15$ m. The resulting values of $Re_l$ and $Ri$ for each hour of the canonical day are now plotted along with the theoretical stability boundary (Fig. 18a).

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5 Wavelength estimates are biased high by the wavelength limit (criterion 3). Assuming the wavelength/shear layer thickness ratio characteristic of KH instability, we estimate this bias as 50%. Reducing the median wavelength by this factor would reduce values of $Re_l$ by a factor of 2. We elect not to do this, because it would add an extra element of complexity and would not alter our conclusions.
Successive points \((\text{Re}_\lambda, \text{Ri})\) trace a clockwise circuit that straddles the stability boundary. At points lying inside (outside) the stability boundary, a hypothetical disturbance with wavelength 50 m would (would not) grow.

We interpret this circuit as follows, starting from the upper left. In early morning, \(\text{Ri}\) is large and \(\text{Re}_\lambda\) is small (0600 LT in Fig. 18a). Large \(\text{Ri}\) indicates that turbulence has mixed out the shear at this depth, while small \(\text{Re}_\lambda\) tells us that some turbulence remains (see Figs. 4e and 17c). After sunrise, stable stratification increases, allowing the wind to accelerate a strong shear flow without instability (slight reduction of \(\text{Ri}\) from 0600 to 1300 LT in Fig. 18a). Turbulence left over from the previous night decays, so the flow becomes strongly sheared but relatively laminar (increase of \(\text{Re}_\lambda\)). In the afternoon, solar heating begins to decrease while wind continues to reinforce the surface shear, reducing \(\text{Ri}\) more rapidly. Shortly after 1400 LT, the flow enters the unstable regime. Billows break and turbulence develops, with its attendant eddy viscosity. This reduces \(\text{Re}_\lambda\), moving the point \((\text{Re}_\lambda, \text{Ri})\) leftward toward the stability boundary. By 2100 LT, \((\text{Re}_\lambda, \text{Ri})\) reaches the stability boundary, that is, turbulence becomes strong enough to damp any disturbance at this wavelength. At this stage, the descending stratified shear layer has passed beyond the layer considered here. However, between 2000 and 0200 LT, the point \((\text{Re}_\lambda, \text{Ri})\) oscillates irregularly around the stability boundary before passing back into the stable regime. (We will see a clearer example of this phenomenon below.) Over the remainder of the night and early morning, shear and stratification mix such that the Richardson number increases, but turbulence decays very slowly. By sunrise, the mean flow is weakly sheared but exhibits remnant turbulence, and the cycle begins again.

So far, our discussion of Fig. 18a has concerned only the stability properties of the phase-averaged background flow with respect to a hypothetical normal-mode disturbance. We now turn back to the actual instabilities computed using 30-min-averaged profiles. Shown in Fig. 18a are points corresponding to all unstable modes found with critical levels in the layer \(-25 < z < -15 \text{ m}\). For these, \(\text{Re}_\lambda\) and \(\text{Ri}\) are computed using the mode wavelength and the instantaneous shear, stratification, and eddy viscosity at the critical level. Unlike the points characterizing the phase-averaged mean flow (colored bullets), these points almost all lie within the stability boundary. Moreover, the points describing individual instabilities tend to lie further within the unstable regime than does the circuit describing the phase-averaged flow. This shows that actual instabilities grow at times when the flow is instantaneously more unstable than usual as a result of some combination of higher shear, lower \(N^2\), and/or weaker turbulence. These temporary destabilization events may represent random interactions.
of internal waves, as has often been suggested (e.g., Smyth et al. 2011), but they can also be caused by the arrival of the diurnally descending shear layer, as we demonstrate in the next subsection.

The cycle shown in Fig. 18a can be computed at any depth. We now examine a second example, $z = -56$ m (Fig. 18b). For this calculation, we choose the wavelength $\lambda = 105$ m, the median for unstable modes with $-61 < z < -51$ m. In contrast to the shallower flow shown in Fig. 18a, this mean flow remains within a small region of the $Re_\lambda$–$Ri$ plane, consistent with its relative remoteness from the diurnal surface forcing. This region lies nearly on the theoretical stability boundary, while most of the points describing individual instabilities lie further into the unstable regime, indicating once again that they result from temporary destabilizations of the mean flow.

Figure 18b illustrates the concept of self-organized criticality, in which a combination of forcing and quasi-random perturbations causes a physical system to oscillate irregularly about a critical state. After originating
in physics (Bak et al. 1987), this idea has been developed further in the oceanic context by Thorpe and Liu (2009) under the name “marginal instability.” A similar oscillation is evident at 20 m between 2200 and 0200 LT (Fig. 18a). At deeper depths (not shown), the values of Reₗ and Ri characterizing the phase-averaged flow lie entirely outside the unstable region.

e. Shear instability and the deep cycle

We now address the question of whether the deep cycle of equatorial mixing, in which diurnally varying turbulence is found far below the direct influence of the surface forcing, can be triggered by shear instabilities. Does diurnally varying instability reach deep enough, and peak at the right time, to account for the observed mixing cycle? On different nights, the bottom of the deep cycle is found anywhere between 50-m depth and the core of the undercurrent (Fig. 3d). In the diurnal phase average (Fig. 4e), turbulence disconnected from surface forcing begins at about 15 m, reaches a maximum at 35 m, and dies away rapidly below 50 m. We have seen that the diurnal cycle of shear instability is clearly visible in the layer $-25 < z < -15$ m (Fig. 17b).

Deeper than this, the cycle is more difficult to discern, and it is not evident at all in the aggregate of modes below $z = -50$ m (Fig. 15c).

For a more detailed view, we divide the upper ocean into six 10-m layers beginning at $z = -15$ m and extending to $z = -75$ m. In addition, we divide the diurnal cycle into eight 3-h intervals. We then compute, for each layer, the fraction of the unstable modes that lie within each 3-h interval, and look for a diurnal cycle in that fraction. Each of the upper four depth ranges shows a clear diurnal cycle (blue, green, red, and cyan curves in Fig. 19a). In the final two depth ranges ($-75 < z < -65$ m and $-65 < z < -55$ m, shown by thin curves), the diurnal cycle is not evident. Thus, the diurnal cycle of shear instability reaches deep enough to coincide with all but the deepest reaches of the deep cycle layer.

We have already seen that, in the layer $-25 < z < -15$ m, instability varies diurnally such that the maximum in the mode count coincides with the shear maximum, and therefore occurs at the right time to initiate the disturbances that will become strong turbulence a few hours later (Fig. 17). We now ask whether the same is true in the deeper layers. In each of the layers where the diurnal cycle of shear instability is found, the maximum mode fraction coincides approximately

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6 This coarsening of the time resolution is necessitated by the finite size of our collection of instabilities. Differences between finer subdivisions of the diurnal cycle would not be statistically significant.
with the passage of the descending shear maximum (Figs. 19a,b,d), though the maxima in the second and third bins (red and green) fall within the same 3-h time bin and thus cannot be distinguished. As we noted in section 2c, the shear maximum precedes the maximum turbulent dissipation (Fig. 4b and e), consistent with the need for instabilities to grow and break before becoming dissipative. We now see that the maximum mode fraction also precedes the maximum dissipation, consistent with the same hypothesis (Figs. 19a,c,d). For $z \gtrsim 230$ m, the dissipation maximum occurs during the same hour even though the instability maximum descends at a finite speed. This is expected, because instabilities growing in the stronger shear found at deeper depths (Fig. 19b) grow and break more rapidly.

Taken together, these results show that the timing of the diurnal instability cycle is consistent with the observed mixing cycle in all layers where it is observed, that is, to well below the maximum of the deep cycle. On this basis, we propose that deep-cycle turbulence is triggered locally by instabilities of the descending stratified shear layer as it enters the marginally unstable environment of the Equatorial Undercurrent.

5. Discussion

Our objective here has been to elucidate the role of shear instability in the diurnal cycles, both shallow and deep, of equatorial mixing. In the instability model of Liu et al. (2012), the growth of a normal-mode perturbation is affected (often reduced or suppressed entirely) by turbulence on scales much smaller than itself via the action of eddy viscosity and diffusivity. This effect has been quantified for the idealized case of a hyperbolic tangent shear layer. Additional models in which the mixing coefficients vary with $z$ show how instability growth can be more or less sensitive to turbulence (Figs. 6 and 14).

Examples of billows growing in the presence of ambient turbulence are plentiful in naturally occurring flows (Smyth and Moum 2012). Examples from the stable regime are harder to identify, as they would require measurements of low Richardson and/or high Reynolds number together with the capacity to demonstrate that billows are not present. Examples have been produced, however, in the direct numerical simulations of Brucker and Sarkar (2007). Although the initial Richardson number is less than $1/4$, and turbulent energy grows over time, visualizations of the initial evolution show no evidence of coherent billows. The turbulent Reynolds number was not computed in that study, but we may form an estimate using their stated value of the peak initial turbulent kinetic energy, $0.12 \mu^2$ by our definition (2). Using the square root of this as a velocity scale and the layer half-thickness as a length scale, we can estimate the eddy viscosity (e.g., Tennekes and Lumley 1972) and hence the equivalent of our turbulent Reynolds number $Re_\lambda$. This turns out to be on the order of unity—sufficient to reduce the critical Richardson number to values well
FIG. 19. (a) Number of unstable modes found in each of the 10-m layers shown in the legend and in 3-h time intervals covering the diurnal cycle. In each layer, the mode count is normalized by its total to give the fraction of modes in each 3-h interval. (b) Diurnal phase average of observed $\langle S^2 \rangle$ in six 10-m layers as listed in (a). (c) Measured $\langle e \rangle$ in each layer. Symbols in (b), (c) indicate the max for the four layers where the diurnal instability cycle is found. (d) Times when $\langle S^2 \rangle$, mode fraction, and $\langle e \rangle$ are at max in each depth layer where the diurnal cycle of instability is evident. In the case of mode fraction, horizontal lines indicate the 3-h time bins. Color coding corresponds to frames (a)–(c).
below 1/4 and plausibly accounting for the absence of billows.

Using an 8-day dataset from the equatorial Pacific, we have identified an ensemble of 698 instabilities. These had critical levels located in the sheared zone between the EUC and the surface, and occurred at points in space and time where the shear was increased (or stratification was reduced) locally. This ensemble of instabilities had wavelengths between 25 and 300 m, with a median of 105 m, and growth rates between 1 and 10 h⁻¹, with a median of 2.5 h⁻¹. (Shorter and slower-growing instabilities were excluded for accuracy.) The direction of the wave vector was strongly correlated with that of the vertical shear of the horizontal current, \( \{ U_v, V_z \} \), where curly brackets denote a vector written in component form.

Numerous observational studies have revealed wavelike signals below the diurnal mixed layer with wavelengths 10³–10⁵ m, which could be expressions of the deeper instabilities found here (e.g., Moum et al. 1992; Lien et al. 1996). Our shallower instabilities (e.g., Fig. 9d) could account for the modes observed by Walsh et al. (1998) and Farrar et al. (2007) in connection with modulations of sea surface temperature.

In many respects, the computed unstable modes resemble the classical KH model of shear instability on a hyperbolic tangent shear layer (Hazel 1972; Maslowe and Thompson 1971). Evidence of this correspondence has been presented previously by Moum et al. (2011) and Smyth et al. (2011). Here, we have seen in addition that the vertical structure of a typical growing mode is similar to that of KH instability (Figs. 8 and 10). Moreover, when plotted against appropriately defined Richardson and Reynolds numbers, nearly all modes lay within the stability boundary for a hyperbolic tangent shear layer with uniform and equal viscosity and diffusivity (Fig. 14b). (This is equally true of some models of nonuniform viscosity and diffusivity; see Fig. 7). Nearly all growth rates are equal to or less than the growth rate of KH modes in an inviscid flow (Fig. 14a), consistent with the damping of instabilities by turbulence.

For the bulk of cases examined here, the effect of nonzero eddy viscosity and diffusivity is to stabilize the flow. However, recent analysis of simple shear layer models with \( \text{Pr}_t \neq 1 \) (Thorpe et al. 2013) indicates that the situation is more complex than this. In some cases, addition of weak viscosity or diffusivity can actually destabilize the flow, so that the critical Richardson number at large (but finite) Re is greater than 1/4. That destabilizing effect could conceivably account for the small fraction of computed modes that lie outside the predicted stability boundary (Fig. 14b).

Our computed instability events follow a diurnal cycle that leads the observed cycle of turbulence (e.g., Moum et al. 1989) by a few hours, a fact that we attribute to the time taken for instabilities to become turbulent. Instability is suppressed during the night, when turbulence is strongest. By plotting the trajectory of a hypothetical normal-mode perturbation on a Reynolds–Richardson number phase plane, we have explored the relationship between the diurnal cycles of instability and turbulence (Figs. 18a,b). In late afternoon and evening, instabilities on a descending, wind-driven shear layer grow, break, and become turbulent. Although the Richardson number remains less than 1/4 throughout the night, turbulence prevents the growth of new normal-mode instabilities. In the early morning, turbulence decays. The converging solar heat flux stabilizes the upper ocean, so that the momentum flux from the wind is convergent, a new shear layer develops, and the cycle begins again. This consistency between the predicted diurnally varying instabilities and the observed mixing pattern lends support to the Liu et al. (2012) theory of instability–turbulence interactions used in our analyses.

The source of the deep cycle of equatorial turbulence has been the object of considerable research since its discovery over a quarter century ago (Moum and Caldwell 1985; Gregg et al. 1985). The present analyses suggest the following scenario. It is typical of the upper-equatorial ocean that both shear and stratification vary more-or-less in proportion so that Ri remains near its critical value. At its upper boundary, this marginally unstable flow is forced by steady winds and a diurnally varying buoyancy flux. The wind provides a continuous supply of kinetic energy to the near-surface current. During the day, penetrating solar radiation stratifies the upper \( \sim 10 \) m, allowing the surface current to increase while remaining dynamically stable. In the afternoon, the surface buoyancy flux diminishes and at some point stability is lost. The accumulated kinetic energy now propagates down through the water column in the form of a stratified shear layer that, through the action of shear instability, entrains the fluid below it and thereby continues its descent.

As the shear layer passes each depth, it tips the ambient flow into the unstable regime, leaving instability and turbulence in its wake. Eventually, the descending shear layer merges with the Equatorial Undercurrent, where it initiates the mixing that we identify as the deep cycle. Near midnight, the shear layer finally dissipates, but the resulting turbulence, supplemented by the kinetic energy of the sheared ambient flow, persists for several hours. Through this process, kinetic energy from the wind is periodically released into the upper ocean, causing turbulent mixing from the surface to the EUC core.

The scenario sketched above is similar to that of Pham et al. (2012, 2013) in that it is initiated with the evening
destabilization of the wind-driven surface current. It differs, however, in that the deep cycle is triggered by the linear amplification of EUC shear by the descending shear layer, as opposed to the nonlinear interaction of EUC shear with hairpin vortices. Note that neither scenario involves the random breaking of gravity waves that has often been assumed (e.g., Smyth et al. 2011). Identification of the relative contributions of these mechanisms, and any others that may exist, remains a problem for future research.

Our description of equatorial mixing as resulting from shear instability may be broadly applicable, or it may be specific to the regime described here, that is, the Pacific cold tongue during a highly energetic La Niña phase. Observations must be extended both in time and in space to determine which view (if either) is most useful over the wide range of conditions that characterize the equatorial oceans.

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REFERENCES
